JETS AND CONNECTIONS IN COMMUTATIVE AND NONCOMMUTATIVE GEOMETRY

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It is emphasized that equivalent definitions of connections on modules over a commutative ring are not so in noncommutative geometry.

1 Introduction

The jet modules $\mathfrak{J}^k(P)$ of a module P over a commutative ring $\mathcal{A}$ are well-known to be a representative object of linear differential operator on P [1]. Furthermore, a connection on a module $\mathcal{A}$ is defined to be a splitting of the exact sequence

$$0 \longrightarrow \mathfrak{O}^1 \otimes P \to \mathfrak{J}^1(P) \xrightarrow{\pi^1_0} P \longrightarrow 0, \tag{1}$$

where $\mathfrak{O}^1$ is the module of differentials of $\mathcal{A}$. In the case of structure modules of smooth vector bundles, these notions of jets and connections coincide with those in differential geometry of fibre bundles where connections on a fibre bundle $Y \to X$ are sections of the affine jet bundle $J^1Y \to Y$ [2]. In general, the notion of jets of modules fails to be extended to modules over a noncommutative ring $\mathcal{A}$ since it implies a certain commutativity property of a differential calculus $\mathfrak{O}^*$ over $\mathcal{A}$. In relation to this circumstance, we match different definitions of connections which being equivalent for modules over a commutative ring are not so in noncommutative geometry.

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2 Modules in noncommutative geometry

Let $\mathcal{A}$ be an associative unital algebra over a commutative ring $\mathcal{K}$, i.e., a $\mathcal{A}$ is a $\mathcal{K}$-ring. One considers right [left] $\mathcal{A}$-modules and $\mathcal{A}$-bimodules (or $\mathcal{A}-\mathcal{A}$-bimodules in the terminology of [3]). A bimodule P over an algebra $\mathcal{A}$ is called a central bimodule if

$$pa = ap, \qquad \forall p \in P, \qquad \forall a \in \mathcal{Z}(\mathcal{A}), \tag{2}$$

where $\mathcal{Z}(\mathcal{A})$ is the centre of the algebra $\mathcal{A}$. By a centre of a $\mathcal{A}$-bimodile P is called a $\mathcal{K}$-submodule $\mathcal{Z}(P)$ of P such that

$$pa \overset{\text{def}}{=} ap, \qquad \forall p \in \mathcal{Z}(P), \qquad \forall a \in \mathcal{A}.$$

If $\mathcal{A}$ is a commutative algebra, every right [left] module P over $\mathcal{A}$ becomes canonically a central bimodule by putting

$$pa = ap, \qquad \forall p \in P, \qquad \forall a \in \mathcal{A}.$$

If $\mathcal{A}$ is a noncommutative algebra, every right [left] $\mathcal{A}$-module P is also a $\mathcal{Z}(\mathcal{A})-\mathcal{A}$-bimodule [$\mathcal{A}-\mathcal{Z}(\mathcal{A})$-bimodule] such that the equality (2) takes place, i.e., it is a central $\mathcal{Z}(\mathcal{A})$-bimodule. From now on, by a $\mathcal{Z}(\mathcal{A})$-bimodule is meant a central $\mathcal{Z}(\mathcal{A})$-bimodule. For the sake of brevity, we say that, given an associative algebra $\mathcal{A}$, right and left $\mathcal{A}$-modules, central $\mathcal{A}$-bimodules and $\mathcal{Z}(\mathcal{A})$-modules are A-modules of type $(1,0)$, $(0,1)$, $(1,1)$ and $(0,0)$, respectively, where $A_0 = \mathcal{Z}(\mathcal{A})$ and $A_1 = \mathcal{A}$. Using this notation, let us recall a few basic operations with modules.

- If P and P' are A-modules of the same type (i,j), so is its direct sum $P \oplus P'$.
- Let P and P' be A-modules of types (i,k) and (k,j), respectively. Their tensor product $P \otimes P'$ (see [3]) defines an A-module of type (i,j).
- Given an A-module P of type (i,j), let $P^* = \mathrm{Hom}_{A_i - A_j}(P, \mathcal{A})$ be its $\mathcal{A}$-dual. One can show that P^* is the module of type $(i+1, j+1) \bmod 2$ [4]. In particular, P and P^{**} are A-modules of the same type. There is the natural homomorphism $P \to P^{**}$. For instance, if P is a projective module of finite rank, so is its dual P^* and $P \to P^{**}$ is an isomorphism [3].

There are several equivalent definitions of a projective module. One says that a right [left] module P is projective if P is a direct summand of a right [left] free module, i.e., there exists a module Q such that $P \oplus Q$ is a free module [3]. Accordingly, a module P is projective if and only if $P = \mathbf{p}S$ where S is a free module and $\mathbf{p}$ is an idempotent, i.e., an endomorphism of S such that $\mathbf{p}^2 = \mathbf{p}$. We will refer to projective $\mathbb{C}^\infty(X)$-modules of finite rank in connection with the Serre–Swan theorem below. Recall that a module is said to be of finite rank or simply finite if it is a quotient of a finitely generated free module.

Noncommutative geometry deals with unital complex involutive algebras (i.e., unital $*$-algebras) as a rule. Let $\mathcal{A}$ be such an algebra (see [5]). It should be emphasized that one cannot use right or left $\mathcal{A}$-modules, but only modules of type $(1,1)$ and $(0,0)$ since the involution of $\mathcal{A}$ reverses the order of product in $\mathcal{A}$. A central $\mathcal{A}$-bimodule P over $\mathcal{A}$ is said to be a $*$-module over a $*$-algebra $\mathcal{A}$ if it is equipped with an antilinear involution $p \mapsto p^*$ such that

$$(apb)^* = b^* p^* a^*, \qquad \forall a, b \in \mathcal{A}, \qquad p \in P.$$

A $*$-module is said to be a finite projective module if it is a finite projective right [left] module.

As well-known, noncommutative geometry is developed in main as a generalization of the calculus in commutative rings of smooth functions. Let X be a locally compact topological space and $\mathcal{A}$ a $*$-algebra $\mathbb{C}^0_0(X)$ of complex continuous functions on X which vanish at infinity of X. Provided with the norm

$$||f|| = \sup_{x\in X} |f|, \qquad f \in \mathcal{A},$$

this algebra is a C^*-algebra [5]. Its spectrum $\widehat{\mathcal{A}}$ is homeomorphic to X. Conversely, any commutative C^*-algebra $\mathcal{A}$ has a locally compact spectrum $\widehat{\mathcal{A}}$ and, in accordance with the well-known Gelfand–Naĭmark theorem, it is isomorphic to the algebra $\mathbb{C}^0_0(\widehat{\mathcal{A}})$ of complex continuous functions on $\widehat{\mathcal{A}}$ which vanish at infinity of $\widehat{\mathcal{A}}$ [5]. If $\mathcal{A}$ is a unital commutative C^*-algebra, its spectrum $\widehat{\mathcal{A}}$ is compact. Let now X be a compact manifold. The $*$-algebra $\mathbb{C}^\infty(X)$ of smooth complex functions on X is a dense subalgebra of the unital C^*-algebra $\mathbb{C}^0(X)$ of continuous functions on X. This is not a C^*-algebra, but it is a Fréchet algebra in its natural locally convex topology of compact convergence for all derivatives. In noncommutative geometry, one does not use the theory of locally convex algebras (see [6]), but considers dense unital subalgebras of C^*-algebras in a purely algebraic fashion.

Let $E \to X$ be a smooth m-dimensional complex vector bundle over a compact manifold X. The module $E(X)$ of its global sections is a $*$-module over the ring $\mathbb{C}^\infty(X)$ of smooth complex functions on X. It is a projective module of finite rank. Indeed, let $(\phi_1, \ldots, \phi_q)$ be a smooth partition of unity such that E is trivial over the sets $U_\zeta \supset \operatorname{supp} \phi_\zeta$, together with the transition functions $\rho_{\zeta\xi}$. Then $p_{\zeta\xi} = \phi_\zeta \rho_{\zeta\xi} \phi_\xi$ are smooth $(m \times m)$-matrix-valued functions on X. They satisfy

$$\sum_\kappa p_{\zeta\kappa} p_{\kappa\xi} = p_{\zeta\xi}, \tag{3}$$

and so assemble into a $(mq \times mq)$-matrix $\mathbf{p}$ whose entries are smooth complex functions on X. Because of (3), we obtain $\mathbf{p}^2 = \mathbf{p}$. Then any section s of $E \to X$ is represented by a column $(\phi_\zeta s^i)$ of smooth complex functions on X such that $\mathbf{p}s = s$. It follows that $s \in \mathbf{p}\mathbb{C}(X)^{mq}$, i.e., $E(X)$ is a projective module. The above mentioned Serre–Swan theorem [7, 8] provides a converse assertion.

THEOREM 1. Let P be a finite projective $*$-module over $\mathbb{C}^\infty(X)$. There exists a complex smooth vector bundle E over X such that P is isomorphic to the module $E(X)$ of global sections of E. $\square$

In noncommutative geometry, one therefore thinks of a finite projective $*$-module over a dense unital $*$-subalgebra of a C^*-algebra as being a noncommutative vector bundle.

3 Commutative differential calculus

Let us summarize some basic facts on the differential calculus in modules over a commutative $\mathcal{K}$-ring $\mathcal{A}$ [1, 2, 9].

Let P and Q be left $\mathcal{A}$-modules. Right modules are studied in a similar way. The set $\operatorname{Hom}_{\mathcal{K}}(P, Q)$ of $\mathcal{K}$-module homomorphisms of P into Q is endowed with the $\mathcal{A} - \mathcal{A}$-bimodule structure by the left and right multiplications

$$(a\phi)(p) = a\phi(p), \qquad (\phi \star a)(p) = \phi(ap), \qquad a \in \mathcal{A}, \qquad p \in P. \tag{4}$$

However, this is not a central $\mathcal{A}$-bimodule because $a\phi \neq \phi \star a$ in general. Let us denote

$$\delta_a \phi = a\phi - \phi \star a. \tag{5}$$

DEFINITION 2. An element $\Delta \in \mathrm{Hom}_{\mathcal{K}}(P, Q)$ is called an s-order linear differential operator from the $\mathcal{A}$-module P to the $\mathcal{A}$-module Q if

$$\delta_{a_0} \circ \cdots \circ \delta_{a_s} \Delta = 0$$

for arbitrary collections of $s+1$ elements of $\mathcal{A}$. It is also called a Q-valued differential operator on P. □

In particular, a first order linear differential operator Δ obeys the condition

$$\delta_a \circ \delta_b \Delta)(p) = \Delta(abp) - a\Delta(bp) - b\Delta(ap) + ab\Delta(p) = 0 \qquad (6)$$

for all $p \in P$, $b, c \in \mathcal{A}$.

A first order differential operator ∂ from $\mathcal{A}$ to an $\mathcal{A}$-module Q is called the Q-valued derivation of the algebra $\mathcal{A}$ if it obeys the Leibniz rule

$$\partial(aa') = a\partial(a') + a'\partial(a), \qquad \forall a, a' \in \mathcal{A}. \qquad (7)$$

This is a particular condition (6).

Turn now to the modules of jets. Given an $\mathcal{A}$-module P, let us consider the tensor product $\mathcal{A} \underset{\mathcal{K}}{\otimes} P$ of $\mathcal{K}$-modules provided with the left $\mathcal{A}$-module structure

$$b(a \otimes p) \overset{\mathrm{def}}{=} (ba) \otimes p, \qquad \forall b \in \mathcal{A}. \qquad (8)$$

For any $b \in \mathcal{A}$, we introduce the left $\mathcal{A}$-module morphism

$$\delta^b(a \otimes p) = (ba) \otimes p - a \otimes (bp). \qquad (9)$$

Let μ^{k+1} be the submodule of the left $\mathcal{A}$-module $\mathcal{A} \underset{\mathcal{K}}{\otimes} P$ generated by all elements of the type

$$\delta^{b_0} \circ \cdots \circ \delta^{b_k}(\mathbf{1} \otimes p).$$

DEFINITION 3. The k-order jet module of the $\mathcal{A}$-module P is defined to be the quotient $\mathfrak{J}^k(P)$ of $\mathcal{A} \otimes P$ by μ^{k+1}. It is a left $\mathcal{A}$-module with respect to the multiplication

$$b(a \otimes p \,\mathrm{mod}\, \mu^{k+1}) = ba \otimes p \,\mathrm{mod}\, \mu^{k+1}. \qquad (10)$$

□

Besides the left $\mathcal{A}$-module structure induced by (8), the k-order jet module $\mathfrak{J}^k(P)$ also admits the left $\mathcal{A}$-module structure given by the multiplication

$$b \star (a \otimes p \bmod \mu^{k+1}) = a \otimes (bp) \bmod \mu^{k+1}. \tag{11}$$

It is called the $\star$-left module structure. There is the $\star$-left $\mathcal{A}$-module homomorphism

$$J^k : P \to \mathfrak{J}^k(P), \qquad J^k p = \mathbf{1} \otimes p \bmod \mu^{k+1}, \tag{12}$$

such that $\mathfrak{J}^k(P)$ as a left $\mathcal{A}$-module is generated by the elements $J^k p$, $p \in P$. It is readily observed that the homomorphism $\mathfrak{J}^k$ (12) is a k-order differential operator (compare the relation (6) and the relation (13) below).

Remark 1. If P is a $\mathcal{A} - \mathcal{A}$-bimodule, the tensor product $\mathcal{A} \underset{\mathcal{K}}{\otimes} P$ is also provided with the right $\mathcal{A}$-module structure

$$(a \otimes p)b \overset{\text{def}}{=} a \otimes pb, \qquad \forall b \in \mathcal{A},$$

and so is the jet module $\mathfrak{J}^k(P)$:

$$(a \otimes p \bmod \mu^{k+1})b = a \otimes (pb) \bmod \mu^{k+1}.$$

If P is a central bimodule, i.e.,

$$ap = pa, \quad \forall a \in \mathcal{A}, \quad p \in P,$$

the $\star$-left $\mathcal{A}$-module structure (11) is equivalent to the right $\mathcal{A}$-module structure (13). •

The jet modules possess the properties similar to those of jet manifolds. In particular, since $\mu^r \subset \mu^s$, $r > s$, there is the the inverse system of epimorphisms

$$\mathfrak{J}^s(P) \xrightarrow{\pi^s_{s-1}} \mathfrak{J}^{s-1}(P) \longrightarrow \cdots \xrightarrow{\pi^1_0} P.$$

Given the repeated jet module $\mathfrak{J}^s(\mathfrak{J}^k(P))$, there exists the monomorphism $\mathfrak{J}^{s+k}(P) \to \mathfrak{J}^s(\mathfrak{J}^k(P))$.

In particular, the first order jet module $\mathfrak{J}^1(P)$ consists of elements $a \otimes p \bmod \mu^2$, i.e., elements $a \otimes p$ modulo the relations

$$\begin{aligned} &\delta^a \circ \delta^b(\mathbf{1} \otimes p) = \\ &\qquad (\delta_a \circ \delta_b \mathfrak{J}^1)(p) = \mathbf{1} \otimes (abp) - a \otimes (bp) - b \otimes (ap) + ab \otimes p = 0. \end{aligned} \tag{13}$$

The morphism $\pi^1_0 : \mathfrak{J}^1(P) \to P$ reads

$$\pi^1_0 : a \otimes p \bmod \mu^2 \to ap. \tag{14}$$

THEOREM 4. For any differential operator $\Delta \in \mathrm{Diff}_s(P, Q)$ there is a unique homomorphism $\mathfrak{f}^\Delta : \mathfrak{J}^s(P) \to Q$ such that the diagram

$$\begin{tikzcd} P \arrow[r, "J^k"] \arrow[dr, "\Delta"'] & \mathfrak{J}^s(P) \arrow[d, "\mathfrak{f}^\Delta"] \\ & Q \end{tikzcd}$$

is commutative. □

Proof. The proof is based on the following fact [1]. Let $h \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{A} \otimes P, Q)$ and

$$\widehat{a} : P \ni p \to a \otimes p \in \mathcal{A} \otimes P,$$

then

$$\delta_b(h \circ \widehat{a})(p) = h(\delta^b(a \otimes p)).$$

QED

The correspondence $\Delta \mapsto \mathfrak{f}^\Delta$ defines the isomorphism

$$\mathrm{Hom}_{\mathcal{A}}(\mathfrak{J}^s(P), Q) = \mathrm{Diff}_s(P, Q), \tag{15}$$

which shows that the jet module $\mathfrak{J}^s(P)$ is the representative object of the functor $Q \to \mathrm{Diff}_s(P, Q)$.

Let us consider the particular jet modules $\mathfrak{J}^s(\mathcal{A})$ of the algebra $\mathcal{A}$, denoted simply by $\mathfrak{J}^s$. The module $\mathfrak{J}^s$ can be provided with the structure of a commutative algebra with respect to the multiplication

$$(aJ^s b) \cdot (a' J^s b) = aa' J^s(bb').$$

For instance, the algebra $\mathfrak{J}^1$ consists of the elements $a \otimes b$ modulo the relations

$$a \otimes b + b \otimes a = ab \otimes \mathbf{1} + \mathbf{1} \otimes ab. \tag{16}$$

It has the left $\mathcal{A}$-module structure

$$c((a \otimes b) \bmod \mu^2) = (ca) \otimes b \bmod \mu^2 \tag{17}$$

(10) and the $\star$-left $\mathcal{A}$-module structure

$$c\star((a\otimes b)\,\mathrm{mod}\,\mu^2)=a\otimes(cb)\,\mathrm{mod}\,\mu^2 \tag{18}$$

(11) which coincides with the right $\mathcal{A}$-module structure (13). We have the canonical monomorphism of left $\mathcal{A}$-modules

$$i_1:\mathcal{A}\to\mathfrak{J}^1, \qquad i_1:a\mapsto a\otimes\mathbf{1}\,\mathrm{mod}\,\mu^2, \tag{19}$$

and the corresponding projection

$$\begin{aligned}&\mathfrak{J}^1\to\mathfrak{J}^1/\mathrm{Im}\,i_1=(\mathrm{Ker}\,\mu^1)\mathrm{mod}\,\mu^2=\mathfrak{O}^1,\\ &a\otimes b\,\mathrm{mod}\,\mu^2\to(a\otimes b-ab\otimes\mathbf{1})\,\mathrm{mod}\,\mu^2.\end{aligned} \tag{20}$$

The quotient $\mathfrak{O}^1$ (20) consists of the elements

$$(a\otimes b-ab\otimes\mathbf{1})\,\mathrm{mod}\,\mu^2, \qquad \forall a,b\in\mathcal{A}.$$

It is provided both with the central $\mathcal{A}$-bimodule structure

$$c((a\otimes b-ab\otimes\mathbf{1})\,\mathrm{mod}\,\mu^2)=(ca\otimes b-cab\otimes\mathbf{1})\,\mathrm{mod}\,\mu^2, \tag{21}$$

$$((\mathbf{1}\otimes ab-b\otimes a)\,\mathrm{mod}\,\mu^2)c=(\mathbf{1}\otimes abc-b\otimes ac)\,\mathrm{mod}\,\mu^2 \tag{22}$$

and the $\star$-left $\mathcal{A}$-module structure

$$c\star((a\otimes b-ab\otimes\mathbf{1})\,\mathrm{mod}\,\mu^2)=(a\otimes cb-acb\otimes\mathbf{1})\,\mathrm{mod}\,\mu^2. \tag{23}$$

It is readily observed that the projection (20) is both the left and $\star$-left module morphisms. Then we have the $\star$-left module morphism

$$\begin{aligned}&d^1:\mathcal{A}\xrightarrow{J^1}\mathfrak{J}^1\to\mathfrak{O}^1,\\ &d^1:b\to\mathbf{1}\otimes b\,\mathrm{mod}\,\mu^2\to(\mathbf{1}\otimes b-b\otimes\mathbf{1})\,\mathrm{mod}\,\mu^2,\end{aligned} \tag{24}$$

such that the central $\mathcal{A}$-bimodule $\mathfrak{O}^1$ is generated by the elements $d^1(b)$, $b\in\mathcal{A}$, in accordance with the law

$$ad^1b=(a\otimes b-ab\otimes\mathbf{1})\,\mathrm{mod}\,\mu^2=(\mathbf{1}\otimes ab)-b\otimes a)\,\mathrm{mod}\,\mu^2=(d^1b)a. \tag{25}$$

PROPOSITION 5. The morphism d^1 (24) is a derivation from $\mathcal{A}$ to $\mathfrak{O}^1$ seen both as a left $\mathcal{A}$-module and $\mathcal{A}$-bimodule. □

Proof. Using the relations (16), one obtains in an explicit form that

$$\begin{aligned} & d^1(ba) = (\mathbf{1}\otimes ba - ba\otimes\mathbf{1}) \bmod \mu^2 = \\ & \qquad (b\otimes a + a\otimes b - ba\otimes\mathbf{1} - ab\otimes\mathbf{1}) \bmod \mu^2 = bd^1a + ad^1b. \end{aligned} \tag{26}$$

This is a $\mathfrak{O}^1$-valued first order differential operator. At the same time,

$$d^1(ba) = (\mathbf{1}\otimes ba - ba\otimes\mathbf{1} + b\otimes a - b\otimes a) \bmod \mu^2 = (d^1b)a + bd^1a.$$

QED

With the derivation d^1 (24), we get the left and $\star$-left module splitting

$$\mathfrak{J}^1 = \mathcal{A}\oplus\mathfrak{O}^1, \tag{27}$$

$$a\mathfrak{J}^1(cb) = ai_1(cb) + ad^1(cb). \tag{28}$$

Accordingly, there is the exact sequence

$$0\to\mathfrak{O}^1\to\mathfrak{J}^1\to\mathcal{A}\to 0 \tag{29}$$

which is split by the monomorphism (19).

PROPOSITION 6. There is the isomorphism

$$\mathfrak{J}^1(P) = \mathfrak{J}^1\otimes P, \tag{30}$$

where by $\mathfrak{J}^1\otimes P$ is meant the tensor product of the right ($\star$-left) $\mathcal{A}$-module $\mathfrak{J}^1$ (18) and the left $\mathcal{A}$-module P, i.e.,

$$[a\otimes b \bmod \mu^2]\otimes p = [a\otimes\mathbf{1} \bmod \mu^2]\otimes bp.$$

□

Proof. The isomorphism (30) is given by the assignment

$$(a\otimes bp) \bmod \mu^2 \leftrightarrow [a\otimes b \bmod \mu^2]\otimes p. \tag{31}$$

QED

The isomorphism (27) leads to the isomorphism

$$\begin{aligned} & \mathfrak{J}^1(P) = (\mathcal{A}\oplus\mathfrak{O}^1)\otimes P, \\ & (a\otimes bp) \bmod \mu^2 \leftrightarrow [(ab + ad^1(b)) \bmod \mu^2]\otimes p, \end{aligned}$$

and to the splitting of left and $\star$-left $\mathcal{A}$-modules

$$\mathfrak{J}^1(P) = (\mathcal{A} \otimes P) \oplus (\mathfrak{O}^1 \otimes P), \tag{32}$$

Applying the projection π^1_0 (14) to the splitting (32), we obtain the exact sequence of left and $\star$-left $\mathcal{A}$-modules (1)

$$\begin{aligned} &0 \to [(a \otimes b - ab \otimes \mathbf{1}) \operatorname{mod} \mu^2] \otimes p \to [(c \otimes \mathbf{1} + a \otimes b - ab \otimes \mathbf{1}) \operatorname{mod} \mu^2] \otimes p \\ &\qquad = (c \otimes p + a \otimes bp - ab \otimes p) \operatorname{mod} \mu^2 \to cp, \end{aligned}$$

similar to the exact sequence (29). This exact sequence has the canonical splitting by the $\star$-left $\mathcal{A}$-module morphism

$$P \ni ap \mapsto a \otimes p + d^1(a) \otimes p.$$

However, the exact sequence (1) needs not be split by a left $\mathcal{A}$-module morphism. Its splitting by a left $\mathcal{A}$-module morphism (see (40) below) implies a connection. On can treat the canonical splitting (19) of the exact sequence (29) as being the canonical connection on the algebra $\mathcal{A}$.

In the case of $\mathfrak{J}^s$, the isomorphism (15) takes the form

$$\operatorname{Hom}_{\mathcal{A}}(\mathfrak{J}^s, Q) = \operatorname{Diff}_s(\mathcal{A}, Q). \tag{33}$$

Then Theorem 4 and Proposition 5 lead to the isomorphism

$$\operatorname{Hom}_{\mathcal{A}}(\mathfrak{O}^1, Q) = \mathfrak{d}(\mathcal{A}, Q). \tag{34}$$

In other words, any Q-valued derivation of $\mathcal{A}$ is represented by the composition $h \circ d^1$, $h \in \operatorname{Hom}_{\mathcal{A}}(\mathfrak{O}^1, Q)$, due to the property $d^1(\mathbf{1}) = 0$.

For instance, if $Q = \mathcal{A}$, the isomorphism (34) reduces to the duality relation

$$\begin{aligned} &\operatorname{Hom}_{\mathcal{A}}(\mathfrak{O}^1, \mathcal{A}) = \mathfrak{d}(\mathcal{A}), \\ &u(a) = u(d^1 a), \qquad a \in \mathcal{A}, \end{aligned} \tag{35}$$

i.e., the module $\mathfrak{d}\mathcal{A}$ coincides with the left $\mathcal{A}$-dual $\mathfrak{O}^{1*}$ of $\mathfrak{O}^1$.

Let us define the modules $\mathfrak{O}^k$ as the skew tensor products of the $\mathcal{K}$-modules $\mathfrak{O}^1$.

PROPOSITION 7. [1]. There are the isomorphisms

$$\operatorname{Hom}_{\mathcal{A}}(\mathfrak{O}^k, Q) = \mathfrak{d}_k(\mathcal{A}, Q), \tag{36}$$

$$\operatorname{Hom}_{\mathcal{A}}(\mathfrak{J}^1(\mathfrak{O}^k), Q) = \mathfrak{d}_k(\operatorname{Diff}_1(Q)). \tag{37}$$

□

The isomorphism (36) is the higher order extension of the isomorphism (34). It shows that the module $\mathfrak{O}^k$ is a representative object of the derivation functor $Q \to \mathfrak{d}_k(\mathcal{A}, Q)$.

The isomorphism (37) implies the homomorphism

$$h^k : \mathfrak{J}^1(\mathfrak{O}^{k-1}) \to \mathfrak{O}^k$$

and defines the operators of exterior differentiation

$$d^k = h^k \circ J^1 : \mathfrak{O}^{k-1} \to \mathfrak{O}^k. \tag{38}$$

These operators constitute the De Rham complex

$$0 \longrightarrow \mathcal{A} \xrightarrow{d^1} \mathfrak{O}^1 \xrightarrow{d^2} \cdots \mathfrak{O}^k \xrightarrow{d^{k+1}} \cdots. \tag{39}$$

4 Connections on commutative modules

There are several equivalent definition of connections on modules over a commutative ring.

Definition 8. By a connection on a $\mathcal{A}$-module P is called a left $\mathcal{A}$-module morphism

$$\Gamma : P \to \mathfrak{J}^1(P), \tag{40}$$
$$\Gamma(ap) = a\Gamma(p), \tag{41}$$

which splits the exact sequence (1). □

This splitting reads

$$J^1 p = \Gamma(p) + \nabla^\Gamma(p), \tag{42}$$

where ∇^Γ is the complementary morphism

$$\nabla^\Gamma : P \to \mathfrak{O}^1 \otimes P, \tag{43}$$
$$\nabla^\Gamma(p) = \mathbf{1} \otimes p \bmod \mu^2 - \Gamma(p).$$

This complementary morphism makes the sense of a covariant differential on the module P, but we will follow the tradition to use the terms "covariant differential"

and "connection" on modules synonymously. With the relation (41), we find that ∇^Γ obeys the Leibniz rule

$$\nabla^\Gamma(ap) = da \otimes p + a\nabla^\Gamma(p). \tag{44}$$

Definition 9. By a connection on a $\mathcal{A}$-module P is meant any morphism ∇ (43) which obeys the Leibniz rule (44), i.e., ∇ is a $(\mathfrak{O}^1 \otimes P)$-valued first order differential operator on P. □

In view of Definition (9) and of the isomorphism (32), it is more convenient to rewrite the exact sequence (1) into the form

$$0 \to \mathfrak{O}^1 \otimes P \to (\mathcal{A} \oplus \mathfrak{O}^1) \otimes P \to P \to 0. \tag{45}$$

Then a connection ∇ on P can be defined as a left $\mathcal{A}$-module splitting of this exact sequence.

In the case of the ring $C^\infty(X)$ and a locally free $C^\infty(X)$-module $\mathcal{S}$ of finite rank, there exist the isomorphisms

$$\begin{aligned} &\mathfrak{O}^1(X) = \mathrm{Hom}_{C^\infty(X)}(\mathfrak{d}(C^\infty(X)), C^\infty(X)), \\ &\mathrm{Hom}_{C^\infty(X)}(\mathfrak{d}(C^\infty(X)), \mathcal{S}) = \mathfrak{O}^1(X) \otimes \mathcal{S}. \end{aligned} \tag{46}$$

With these isomorphisms, we come to other equivalent definitions of a connection on modules.

Definition 10. Any morphism

$$\nabla : \mathcal{S} \to \mathrm{Hom}_{C^\infty(X)}(\mathfrak{d}(C^\infty(X)), \mathcal{S}) \tag{47}$$

satisfying the Leibniz rule (44) is called a connection on a $C^\infty(X)$-module $\mathcal{S}$. □

Definition 11. By a connection on a $C^\infty(X)$-module $\mathcal{S}$ is meant a $C^\infty(X)$-module morphism

$$\mathfrak{d}(C^\infty(X)) \ni \tau \mapsto \nabla_\tau \in \mathrm{Diff}_1(\mathcal{S}, \mathcal{S}) \tag{48}$$

such that the first order differential operators ∇_τ obey the rule

$$\nabla_\tau(fs) = (\tau\rfloor df)s + f\nabla_\tau s. \tag{49}$$

□

If a $\mathcal{S}$ is a commutative $C^\infty(X)$-ring, Definition 11 can be modified as follows.

DEFINITION 12. By a connection on $C^\infty(X)$-ring $\mathcal{S}$ is meant any $C^\infty(X)$-module morphism

$$\mathfrak{d}(C^\infty(X)) \ni \tau \mapsto \nabla_\tau \in \mathfrak{d}\mathcal{S} \tag{50}$$

which is a connection on $\mathcal{S}$ as a $C^\infty(X)$-module, i.e., obeys the Leinbniz rule (49). □

Two such connections ∇_τ and ∇'_τ differ from each other in a derivation of the ring $\mathcal{S}$ which vanishes on $C^\infty(X) \subset \mathcal{S}$.

5 Noncommutative differential calculus

One believes that a noncommutative generalization of differential geometry should be given by a $\mathbb{Z}$-graded differential algebra which replaces the exterior algebra of differential forms [10]. This viewpoint is more general than that implicit above where a noncommutative ring replaces a ring of smooth functions.

Recall that a graded algebra Ω^* over a commutative ring $\mathcal{K}$ is defined as a direct sum

$$\Omega^* = \mathop{\oplus}_{k=0} \Omega^k$$

of $\mathcal{K}$-modules Ω^k, provided with the associative multiplication law such that $\alpha \cdot \beta \in \Omega^{|\alpha|+|\beta|}$, where $|\alpha|$ denotes the degree of an element $\alpha \in \Omega^{|\alpha|}$. In particular, Ω^0 is a unital $\mathcal{K}$-algebra $\mathcal{A}$, while $\Omega^{k>0}$ are $\mathcal{A}$-bimodules. A graded algebra Ω^* is called a graded differential algebra if it is a cochain complex of $\mathcal{K}$-modules

$$0 \longrightarrow \mathcal{A} \xrightarrow{\delta} \Omega^1 \xrightarrow{\delta} \cdots$$

with respect to a coboundary operator δ such that

$$\delta(\alpha \cdot \beta) = \delta\alpha \cdot \beta + (-1)^{|\alpha|}\alpha \cdot \delta\beta.$$

A graded differential algebra (Ω^*, δ) with $\Omega^0 = \mathcal{A}$ is called the differential calculus over $\mathcal{A}$. If $\mathcal{A}$ is a $*$-algebra, we have additional conditions

$$(\alpha \cdot \beta)^* = (-1)^{|\alpha||\beta|}\beta^*\alpha^*,$$
$$(\delta\alpha)^* = \delta(\alpha^*).$$

Remark 2. The De Rham complex (39) exemplifies a differential calculus over a commutative ring. To generalize it to a noncommutative ring $\mathcal{A}$, the coboundary operator δ should have the additional properties:

- $\Omega^{k>0}$ are central $\mathcal{A}$-bimodules,
- elements $\delta a_1 \cdots \delta a_k$, $a_i \in \mathcal{Z}(\mathcal{A})$, belong to the centre $\mathcal{Z}(\Omega^k)$ of the module Ω^k. Then, if $\mathcal{A}$ is a commutative ring, the commutativity condition (25) holds.

•

Let $\Omega^*\mathcal{A}$ be the smallest differential subalgebra of the algebra Ω^* which contains $\mathcal{A}$. As an $\mathcal{A}$-algebra, it is generated by the elements δa, $a \in \mathcal{A}$, and consists of finite linear combinations of monomials of the form

$$\alpha = a_0\delta a_1 \cdots \delta a_k, \qquad a_i \in \mathcal{A}. \tag{51}$$

The product of monomials (51) is defined by the rule

$$(a_0\delta a_1) \cdot (b_0\delta b_1) = a_0\delta(a_1 b_0) \cdot \delta b_1 - a_0 a_1 \delta b_0 \cdot \delta b_1.$$

In particular, $\Omega^1\mathcal{A}$ is a $\mathcal{A}$-bimodule generated by elements δa, $a \in A$. Because of

$$(\delta a)b = \delta(ab) - a\delta b,$$

the bimodule $\Omega^1\mathcal{A}$ can also be seen as a left [right] $\mathcal{A}$-module generated by the elements δa, $a \in \mathcal{A}$. Note that $\delta(\mathbf{1}) = 0$. Accordingly,

$$\Omega^k\mathcal{A} = \underbrace{\Omega^1\mathcal{A} \cdots \Omega^1\mathcal{A}}_{k}$$

are $\mathcal{A}$-bimodules and, simultaneously, left [right] $\mathcal{A}$-modules generated by monomials (51).

The differential subalgebra $(\Omega^*\mathcal{A}, \delta)$ is a differential calculus over $\mathcal{A}$. It is called the universal differential calculus because of the following property [11, 12, 13].

Let (Ω'^*, δ') be another differential calculus over a unital $\mathcal{K}$-algebra $\mathcal{A}'$, and let $\rho : \mathcal{A} \to \mathcal{A}'$ be an algebra morphism. There exists a unique extension of this morphism to a morphism of graded differential algebras

$$\rho^k : \Omega^k\mathcal{A} \to \Omega'^k$$

such that $\rho^{k+1} \circ \delta = \delta' \circ \rho^k$.

Our interest to differential calculi over an algebra $\mathcal{A}$ is caused by the fact that, in commutative geometry, Definition 9 of a connection on an $\mathcal{A}$-module requires the module $\mathfrak{O}^1$ (20). If $\mathcal{A} = C^\infty(X)$, this module is the module of 1-forms on X. To introduce connections in noncommutative geometry, one therefore should construct the noncommutative version of the module $\mathfrak{O}^1$. We may follow the construction of $\mathfrak{O}^1$ in Section 3, but not take the quotient by modμ^2 that implies the commutativity condition (25).

Remark 3. This is the crucial poin that does not enable us to generalize the notion of jets of modules to modules over a noncommutative ring unless the very particular case when $d\mathcal{A}$ belongs to the centre of the module Ω^1. •

Given a unital $\mathcal{K}$-algebra $\mathcal{A}$, let us consider the tensor product $\mathcal{A}\underset{\mathcal{K}}{\otimes}\mathcal{A}$ of $\mathcal{K}$-modules and the $\mathcal{K}$-module morphism

$$\mu^1 : \mathcal{A}\underset{\mathcal{K}}{\otimes}\mathcal{A} \ni a \otimes b \mapsto ab \in \mathcal{A}.$$

Following (20), we define the $\mathcal{K}$-module

$$\overline{\mathfrak{O}}^1[\mathcal{A}] = \mathrm{Ker}\,\mu^1. \tag{52}$$

There is the $\mathcal{K}$-module morphism

$$d : \mathcal{A} \ni a \mapsto (\mathbf{1} \otimes a - a \otimes \mathbf{1}) \in \overline{\mathfrak{O}}^1[\mathcal{A}] \tag{53}$$

(cf. (24)). Moreover, $\overline{\mathfrak{O}}^1[\mathcal{A}]$ is a $\mathcal{A}$-bimodule generated by the elements da, $a \in A$, with the multiplication law

$$b(da)c = b \otimes ac - ba \otimes c, \qquad a, b, c \in \mathcal{A}.$$

The morphism d (53) possesses the property

$$d(ab) = (\mathbf{1} \otimes ab - ab \otimes \mathbf{1} + a \otimes b - a \otimes b) = (da)b + adb \tag{54}$$

(cf. (26)), i.e., d is a $\overline{\mathfrak{O}}^1[\mathcal{A}]$-valued derivation of $\mathcal{A}$. Due to this property, $\overline{\mathfrak{O}}^1[\mathcal{A}]$ can be seen as a left $\mathcal{A}$-module generated by the elements da, $a \in \mathcal{A}$. At the same time, if $\mathcal{A}$ is a commutative ring, the $\mathcal{A}$-bimodule $\overline{\mathfrak{O}}^1[\mathcal{A}]$ does not coincide with the bimodule $\mathfrak{O}^1$ (20) because $\overline{\mathfrak{O}}^1[\mathcal{A}]$ is not a central bimodule (see Remark 2).

To overcome this difficulty, let us consider the $\mathcal{Z}(\mathcal{A})$ of derivations of the algebra $\mathcal{A}$. They obey the rule

$$u(ab) = u(a)b + au(b), \qquad \forall a, b \in \mathcal{A}. \tag{55}$$

It should be emphasized that the derivation rule (55) differs from that

$$u(ab) = u(a)b + u(b)a$$

for a general algebra [14]. By virtue of (55), derivations of an algebra $\mathcal{A}$ constitute a $\mathcal{Z}(\mathcal{A})$-bimodule, but not a left $\mathcal{A}$-module.

The $\mathcal{Z}(\mathcal{A})$-bimodule $\mathfrak{d}\mathcal{A}$ is also a Lie algebra over the commutative ring $\mathcal{K}$ with respect to the Lie bracket

$$[u, u'] = u \circ u' - u' \circ u. \tag{56}$$

The centre $\mathcal{Z}(\mathcal{A})$ is stable under $\mathfrak{d}\mathcal{A}$, i.e.,

$$u(a)b = bu(a), \quad \forall a \in \mathcal{Z}(\mathcal{A}), \quad b \in \mathcal{A}, \quad u \in \mathfrak{d}\mathcal{A},$$

and one has

$$[u, au'] = u(a)u' + a[u, u'], \quad \forall a \in \mathcal{Z}(\mathcal{A}), \quad u, u' \in \mathfrak{d}\mathcal{A}. \tag{57}$$

If $\mathcal{A}$ is a unital $*$-algebra, the module $\mathfrak{d}\mathcal{A}$ of derivations of $\mathcal{A}$ is provided with the involution $u \mapsto u^*$ defined by

$$u^*(a) = (u(a^*))^*.$$

Then the Lie bracket (56) satisfies the reality condition $[u, u']^* = [u^*, u'^*]$.

Let us consider the Chevalley–Eilenberg cohomology (see [15]) of the Lie algebra $\mathfrak{d}\mathcal{A}$ with respect to its natural representation in $\mathcal{A}$. The corresponding k-cochain space $\underline{\mathfrak{O}}^k[\mathcal{A}]$, $k = 1, \ldots$, is the $\mathcal{A}$-bimodule of $\mathcal{Z}(\mathcal{A})$-multilinear antisymmetric mappings of $\mathfrak{d}\mathcal{A}^k$ to $\mathcal{A}$. In particular, $\underline{\mathfrak{O}}^1[\mathcal{A}]$ is the $\mathcal{A}$-dual

$$\underline{\mathfrak{O}}^1[\mathcal{A}] = \mathfrak{d}\mathcal{A}^* \tag{58}$$

of the derivation module $\mathfrak{d}\mathcal{A}$ (cf. (46)). Put $\underline{\mathfrak{O}}^0[\mathcal{A}] = \mathcal{A}$. The Chevalley–Eilenberg coboundary operator

$$d : \underline{\mathfrak{O}}^k[\mathcal{A}] \to \underline{\mathfrak{O}}^{k+1}[\mathcal{A}]$$

is given by

$$(d\phi)(u_0, \ldots, u_k) = \frac{1}{k+1} \sum_{i=0}^{k} (-1)^i u_i(\phi(u_0, \ldots, \widehat{u_i}, \ldots, u_k)) + \tag{59}$$
$$\frac{1}{k+1} \sum_{0 \le r < s \le k} (-1)^{r+s} \phi([u_r, u_s], u_0, \ldots, \widehat{u_r}, \ldots, \widehat{u_s}, \ldots, u_k),$$

where $\widehat{u_i}$ means omission of u_i. For instance,

$$(da)(u) = u(a), \qquad a \in \mathcal{A}, \tag{60}$$
$$(d\phi)(u_0, u_1) = \frac{1}{2}(u_0(\phi(u_1)) - u_1(\phi(u_0)) - \phi([u_0, u_1])), \quad \phi \in \mathfrak{O}^1[\mathcal{A}]. \tag{61}$$

It is readily observed that $d^2 = 0$, and we have the Chevalley–Eilenberg cochain complex of $\mathcal{K}$-modules

$$0 \longrightarrow \mathcal{A} \xrightarrow{d} \underline{\mathfrak{O}}^k[\mathcal{A}] \xrightarrow{d} \cdots. \tag{62}$$

Furthermore, the $\mathbb{Z}$-graded space

$$\underline{\mathfrak{O}}^*[\mathcal{A}] = \mathop{\oplus}_{k=0} \underline{\mathfrak{O}}^k[\mathcal{A}] \tag{63}$$

is provided with the structure of a graded algebra with respect to the multiplication $\wedge$ combining the product of $\mathcal{A}$ with antisymmetrization in the arguments. Notice that, if $\mathcal{A}$ is not commutative, there is nothing like graded commutativity of forms, i.e.,

$$\phi \wedge \phi' \neq (-1)^{|\phi||\phi'|} \phi' \wedge \phi$$

in general. If $\mathcal{A}$ is a $*$-algebra, $\underline{\mathfrak{O}}^*[\mathcal{A}]$ is also equipped with the involution

$$\phi^*(u_1, \ldots, u_k) \stackrel{\mathrm{def}}{=} (\phi(u_1^*, \ldots, u_k^*))^*.$$

Thus, $(\underline{\mathfrak{O}}^*[\mathcal{A}], d)$ is a differential calculus over $\mathcal{A}$, called the Chevalley–Eilenberg differential calculus.

It is easy to see that, if $\mathcal{A} = \mathbb{C}^\infty(X)$ is the commutative ring of smooth complex functions on a compact manifold X, the graded algebra $\underline{\mathfrak{O}}^*[\mathbb{C}^\infty(X)]$ is exactly the

complexified exterior algebra $\mathbb{C}\otimes\mathfrak{O}^*(X)$ of exterior forms on X. In this case, the coboundary operator (59) coincides with the exterior differential, and (62) is the De Rham complex of complex exterior forms on a manifold X. In particular, the operations

$$(u\rfloor\phi)(u_1,\ldots,u_{k-1}) = k\phi(u,u_1,\ldots,u_{k-1}), \qquad u\in\mathfrak{d}\mathcal{A},$$
$$\mathbf{L}_u(\phi) = d(u\rfloor\phi) + u\rfloor f(\phi),$$

are the noncommutative generalizations of the contraction and the Lie derivative of differential forms. These facts motivate one to think of elements of $\mathfrak{O}^1[\mathcal{A}]$ as being a noncommutative generalization of differential 1-forms, though this generalization by no means is unique.

Let $\mathfrak{O}^*[\mathcal{A}]$ be the smallest differential subalgebra of the algebra $\underline{\mathfrak{O}}^*[\mathcal{A}]$ which contains $\mathcal{A}$. It is generated by the elements da, $a\in\mathcal{A}$, and consists of finite linear combinations of monomials of the form

$$\phi = a_0 da_1\wedge\cdots\wedge da_k, \qquad a_i\in\mathcal{A},$$

(cf. (51)). In particular, $\mathfrak{O}^1[\mathcal{A}]$ is a $\mathcal{A}$-bimodule (52) generated by da, $a\in\mathcal{A}$. Since the centre $\mathcal{Z}(\mathcal{A})$ of $\mathcal{A}$ is stable under derivations of $\mathcal{A}$, we have

$$bda = (da)b, \qquad adb = (db)a, \qquad a\in\mathcal{A}, \qquad b\in\mathcal{Z}(\mathcal{A}),$$
$$da\wedge db = -db\wedge da, \qquad \forall a\in\mathcal{Z}(\mathcal{A}).$$

Hence, $\mathfrak{O}^1[\mathcal{A}]$ is a central bimodule in contrast with the bimodule $\overline{\mathfrak{O}}^1[\mathcal{A}]$ (52). By virtue of the relation (60), we have the isomorphism

$$\mathfrak{d}\mathcal{A} = \mathfrak{O}^1[\mathcal{A}]^* \tag{64}$$

of the $\mathcal{Z}(\mathcal{A})$-module $\mathfrak{d}\mathcal{A}$ of derivations of $\mathcal{A}$ to the $\mathcal{A}$-dual of the module $\mathfrak{O}^1[\mathcal{A}]$ (cf. (35)). Combining the duality relations (58) and (64) gives the relation

$$\underline{\mathfrak{O}}^1[\mathcal{A}] = \mathfrak{O}^1[\mathcal{A}]^{**}.$$

The differential subalgebra $(\mathfrak{O}^*[\mathcal{A}],d)$ is a universal differential calculus over $\mathcal{A}$. If $\mathcal{A}$ is a commutative ring, then $\mathfrak{O}^*[\mathcal{A}]$ is the De Rham complex (39).

6 Universal connections

Let (Ω^*, δ) be a differential calculus over a unital $\mathcal{K}$-algebra $\mathcal{A}$ and P a left [right] $\mathcal{A}$-module. Similarly to Definition 9, one can construct the tensor product $\Omega^1 \otimes P$ $[P \otimes \Omega^1]$ and define a connection on P as follows [8, 13].

DEFINITION 13. A noncommutative connection on a left $\mathcal{A}$-bimodule P with respect to the differential calculus (Ω^*, δ) is a $\mathcal{K}$-module morphism

$$\nabla : P \to \Omega^1 \otimes P \tag{65}$$

which obeys the Leibniz rule

$$\nabla(ap) = \delta a \otimes p + a\nabla(p).$$

□

If $\Omega^* = \Omega^*\mathcal{A}$ is a universal differential calculus, the connection (65) is called a universal connection [8, 13].

The curvature of the noncommutative connection (65) is defined as the $\mathcal{A}$-module morphism

$$\nabla^2 : P \to \mathfrak{O}^2[\mathcal{A}] \otimes P$$

[13]. Note also that the morphism (65) has a natural extension

$$\nabla : \Omega^k \otimes P \to \Omega^{k+1} \otimes P,$$
$$\nabla(\alpha \otimes p) = \delta\alpha \otimes p + (-1)^{|\alpha|}\alpha \otimes \nabla(p), \qquad \alpha \in \Omega^*,$$

[13, 16].

Similarly, a noncommutative connection on a right $\mathcal{A}$-module is defined. However, a connection on a left [right] module does not necessarily exist as it is illustrated by the following theorem.

THEOREM 14. A left [right] universal connection on a left [right] module P of finite rank exists if and only if P is projective [13, 17]. □

The problem arises when P is a $\mathcal{A}$-bimodule. If $\mathcal{A}$ is a commutative ring, left and right module structures of an $\mathcal{A}$-bimodule are equivalent, and one deals with either a left or right noncommutative connection on P (see Definition 9). If P is a

$\mathcal{A}$-bimodule over a noncommutative ring, left and right connections ∇^L and ∇^R on P should be considered simultaneously. However, the pair (∇^L, ∇^R) by no means is a bimodule connection since $\nabla^L(P) \in \Omega^1 \otimes P$, whereas $\nabla^R(P) \in P \otimes \Omega^1$. As a palliative, one assumes that there exists a bimodule isomorphism

$$\varrho : \Omega^1 \otimes P \to P \otimes \Omega^1. \tag{66}$$

Then a pair (∇^L, ∇^R) of right and left noncommutative connections on P is called a ϱ-compatible if

$$\varrho \circ \nabla^L = \nabla^R$$

[13, 16, 18] (see also [19] for a weaker condition). Nevertheless, this is not a true bimodule connection (see the condition (70) below).

Remark 4. If $\mathcal{A}$ is a commutative ring, the isomorphism ϱ (2) is naturally the permutation

$$\varrho : \alpha \otimes p \mapsto p \otimes \alpha, \qquad \forall \alpha \in \Omega^1, \quad p \in P.$$

•

The above mentioned problem of a bimodule connection is not simplified radically even if $P = \Omega^1$, together with the natural permutations

$$\phi \otimes \phi' \mapsto \phi' \otimes \phi, \qquad \phi, \phi' \in \Omega^1,$$

[4, 18].

Let now $(\mathfrak{O}^*[\mathcal{A}], d)$ be the universal differential calculus over a noncommutative $\mathcal{K}$-ring $\mathcal{A}$. Let

$$\begin{aligned} &\nabla^L : P \to \mathfrak{O}^1[\mathcal{A}] \otimes P, \\ &\nabla^L(ap) = da \otimes p + a\nabla^L(p). \end{aligned} \tag{67}$$

be a left universal connection on a left $\mathcal{A}$-module P (cf. Definition 9). Due to the duality relation (64), there is the $\mathcal{K}$-module endomorphism

$$\nabla^L_u : P \ni p \to u \rfloor \nabla^L(p) \in P \tag{68}$$

of P for any derivation $u \in \mathfrak{d}\mathcal{A}$. If ∇^R is a right universal connection on a right $\mathcal{A}$-module P, the similar endomorphism

$$\nabla^R_u : P \ni p \to \nabla^L(p) \lfloor u \in P \tag{69}$$

takes place for any derivation $u \in \mathfrak{d}\mathcal{A}$. Let (∇^L, ∇^R) be a ϱ-compatible pair of left and right universal connections on an $\mathcal{A}$-bimodule P. It seems natural to say that this pair is a bimodule universal connection on P if

$$u\rfloor\nabla^L(p) = \nabla^R(p)\lfloor u \tag{70}$$

for all $p \in P$ and $u \in \mathfrak{d}\mathcal{A}$. Nevertheless, motivated by the endomorphisms (68) – (69), one can suggest another definition of connections on a bimodule, similar to Definition 11.

7 The Dubois-Violette connection

Let $\mathcal{A}$ be $\mathcal{K}$-ring and P an A-module of type (i, j) in accordance with the notation in Section 2.

Definition 15. By analogy with Definition 11, a Dubois-Violette connection on an A-module P of type (i, j) is a $\mathcal{Z}(\mathcal{A})$-bimodule morphism

$$\nabla : \mathfrak{d}\mathcal{A} \ni u \mapsto \nabla_u \in \mathrm{Hom}_{\mathcal{K}}(P, P) \tag{71}$$

of $\mathfrak{d}\mathcal{A}$ to the $\mathcal{Z}(\mathcal{A})$-bimodule of endomorphisms of the $\mathcal{K}$-module P which obey the Leibniz rule

$$\nabla_u(a_i p a_j) = u(a_i)pa_j + a_i\nabla_u(p)a_j + a_i p u(a_j), \quad \forall p \in P, \quad \forall a_k \in A_k, \tag{72}$$

[4, 18]. □

By virtue of the duality relation (64) and the expressions (68) – (69), every left [right] universal connection yields a connection (71) on a left [right] $\mathcal{A}$-module P. From now on, by a connection in noncommutative geometry is meant a Dubois-Violette connection in accordance with Definition (15).

A glance at the expression (72) shows that, if connections on an A-module P of type (i, j) exist, they constitute an affine space modelled over the linear space of $\mathcal{Z}(\mathcal{A})$-bimodule morphisms

$$\sigma : \mathfrak{d}\mathcal{A} \ni u \mapsto \sigma_u \in \mathrm{Hom}_{A_i - A_j}(P, P)$$

of $\mathfrak{d}\mathcal{A}$ to the $\mathcal{Z}(\mathcal{A})$-bimodule of endomorphisms

$$\sigma_u(a_i p a_j) = a_i\sigma(p)a_j, \quad \forall p \in P, \quad \forall a_k \in A_k,$$

of the A-module P.

Example 5. If $P=\mathcal{A}$, the morphisms

$$\nabla_u(a)=u(a), \quad \forall u\in\mathfrak{d}\mathcal{A}, \quad \forall a\in\mathcal{A}, \tag{73}$$

define a canonical connection on $\mathcal{A}$ in accordance with Definition 15. Then the Leibniz rule (72) shows that any connection on a central $\mathcal{A}$-bimodule P is also a connection on P seen as a $\mathcal{Z}(\mathcal{A})$–bimodule. •

Example 6. If P is a $\mathcal{A}$-bimodule and $\mathcal{A}$ has only inner derivations

$$\mathrm{ad}\, b(a)=ba-ab,$$

the morphisms

$$\nabla_{\mathrm{ad}b}(p)=bp-pb, \quad \forall b\in\mathcal{A}, \quad \forall p\in P, \tag{74}$$

define a canonical connection on P. •

By the curvature R of a connection ∇ (71) on an A-module P is meant the $\mathcal{Z}(\mathcal{A})$-module morphism

$$\begin{aligned} &R:\mathfrak{d}\mathcal{A}\times\mathfrak{d}\mathcal{A}\ni(u,u')\to R_{u,u'}\in \mathrm{Hom}\,_{A_i-A_j}(P,P), \\ &R_{u,u'}(p)=\nabla_u(\nabla_{u'}(p))-\nabla_{u'}(\nabla_u(p))-\nabla_{[u,u']}(p), \quad p\in P, \end{aligned} \tag{75}$$

[4]. We have

$$\begin{aligned} &R_{au,a'u'}=aa'R_{u,u'}, \qquad a,a'\in\mathcal{Z}(\mathcal{A}), \\ &R_{u,u'}(a_ipb_j)=a_iR_{u,u'}(p)b_j, \quad a_i\in A_j, \quad b_j\in A_j. \end{aligned}$$

For instance, the curvature of the connections (73) and (74) vanishes.

Let us provide some standard operations with the connections (71).

(i) Given two modules P and P' of the same type (i,j) and connections ∇ and ∇' on them, there is an obvious connection $\nabla\oplus\nabla'$ on $P\oplus P'$.

(ii) Let P be a module of type (i,j) and P^* its $\mathcal{A}$-dual. For any connection ∇ on P, there is a unique dual connection ∇' on P^* such that

$$u(\langle p,p'\rangle)=\langle\nabla_u(p),p'\rangle+\langle p,\nabla'(p')\rangle, \quad p\in P, \quad p'\in P^*, \quad u\in\mathfrak{d}\mathcal{A}.$$

(iii) Let P_1 and P_2 be A-modules of types (i,k) and (k,j), respectively, and let ∇^1 and ∇^2 be connections on these modules. For any $u \in \mathfrak{d}\mathcal{A}$, let us consider the endomorphism

$$(\nabla^1 \otimes \nabla^2)_u = \nabla^1_u \otimes \mathrm{Id}\, P_1 + \mathrm{Id}\, P_2 \otimes \nabla^2_u \tag{76}$$

of the tensor product $P_1 \otimes P_2$ of $\mathcal{K}$-modules P_1 and P_2. This endomorphism preserves the subset of $P_1 \otimes P_2$ generated by elements

$$p_1 a \otimes p_2 - p_1 \otimes a p_2,$$

with $p_1 \in P_1$, $p_2 \in P_2$ and $a \in A_k$. Due to this fact, the endomorphisms (76) define a connection on the tensor product $P_1 \otimes P_2$ of modules P_1 and P_2.

(iv) If $\mathcal{A}$ is a unital $*$-algebra, we have only modules of type $(1,1)$ and $(0,0)$, i.e., $*$-modules and $\mathcal{Z}(\mathcal{A})$-bimodules. Let P be a module of one of these types. If ∇ is a connection on P, there exists a conjugate connection ∇^* on P given by the relation

$$\nabla^*_u(p) = (\nabla_{u^*}(p^*))^*. \tag{77}$$

A connection ∇ on P is said to be real if $\nabla = \nabla^*$.

Let now $P = \underline{\mathfrak{O}}^1[\mathcal{A}]$. A connection on $\mathcal{A}$-bimodule $\underline{\mathfrak{O}}^1[\mathcal{A}]$ is called a linear connection [4, 18]. Note that this is not the term for an arbitrary left [right] connection on $\underline{\mathfrak{O}}^1[\mathcal{A}]$ [16]. If $\underline{\mathfrak{O}}^1[\mathcal{A}]$ is a $*$-module, a linear connection on it is assumed to be real. Given a linear connection ∇ on $\underline{\mathfrak{O}}^1[\mathcal{A}]$, there is a $\mathcal{A}$-bimodule homomorphism, called the torsion of the connection ∇,

$$\begin{aligned} &T : \underline{\mathfrak{O}}^1[\mathcal{A}] \to \underline{\mathfrak{O}}^2[\mathcal{A}], \\ &(T\phi)(u,u') = (d\phi)(u,u') - \nabla_u(\phi)(u') + \nabla_{u'}(\phi)(u), \end{aligned} \tag{78}$$

for all $u, u' \in \mathfrak{d}\mathcal{A}$, $\phi \in \underline{\mathfrak{O}}^1[\mathcal{A}]$.

8 Matrix geometry

This Section gives a standard example of linear connections in matrix geometry when $\mathcal{A} = M_n$ is the algebra of complex $(n \times n)$-matrices [20, 21, 22].

Let $\{\varepsilon_r\}$, $1 \le r \le n^2-1$, be an anti-Hermitiam basis of the Lie algebra $su(n)$. Elements ε_r generate M_n as an algebra, while $u_r = \mathrm{ad}\, \varepsilon_r$ constitute a basis of the right

Lie algebra $\mathfrak{d}M_n$ of derivations of the algebra M_n, together with the commutation relations

$$[u_r, u_q] = c^s_{rq} u_s,$$

where c^s_{rq} are structure constants of the Lie algebra $su(n)$. Since the centre $\mathcal{Z}(M_n)$ of M_n consists of matrices $\lambda\mathbf{1}$, $\mathfrak{d}M_n$ is a complex free module of rank $n^2 - 1$.

Let us consider the universal differential calculus $(\mathfrak{O}^*[M_n], d)$ over the algebra M_n, where d is the Chevalley–Eilenberg coboundary operator (59). There is a convenient system $\{\theta^r\}$ of generators of $\mathfrak{O}^1[M_n]$ seen as a left M_n-module. They are given by the relations

$$\theta^r(u_q) = \delta^r_q \mathbf{1}.$$

Hence, $\mathfrak{O}^1[M_n]$ is a free left M_n-module of rank $n^2 - 1$. It is readily observed that elements θ^r belong to the centre of the M_n-bimodule $\mathfrak{O}^1[M_n]$, i.e.,

$$a\theta^r = \theta^r a, \qquad \forall a \in M_n. \tag{79}$$

It also follows that

$$\theta^r \wedge \theta^q = -\theta^q \wedge \theta^r. \tag{80}$$

The morphism $d : M_n \to \mathfrak{O}^1[M_n]$ is given by the formula (60). It reads

$$d\varepsilon_r(u_q) = \operatorname{ad}\varepsilon_q(\varepsilon_r) = c^s_{qr}\varepsilon_s,$$

that is,

$$d\varepsilon_r = c^s_{qr}\varepsilon_s\theta^q. \tag{81}$$

The formula (61) leads to the Maurer–Cartan equations

$$d\theta^r = -\frac{1}{2}c^r_{qs}\theta^q \wedge \theta^s. \tag{82}$$

If we define $\theta = \varepsilon_r\theta^r$, the equality (81) can be rewritten as

$$da = a\theta - \theta a, \qquad \forall a \in M_n.$$

It follows that the M_n-bimodule $\mathfrak{O}^1[M_n]$ is generated by the element θ. Since $\mathfrak{d}M_n$ is a finite free module, one can show that the M_n-bimodule $\mathfrak{O}^1[M_n]$ is isomorphic to the M_n-dual $\underline{\mathfrak{O}}^1[M_n]$ of $\mathfrak{d}M_n$.

Turn now to connections on the M_n-bimodule $\mathfrak{O}^1[M_n]$. Such a connection ∇ is given by the relations

$$\begin{aligned}
&\nabla_{u=c^r u_r} = c^r \nabla_r, \\
&\nabla_r(\theta^p) = \omega^p_{rq}\theta^q, \qquad \omega^p_{rq} \in M_n.
\end{aligned} \tag{83}$$

Bearing in mind the equalities (79) – (80), we obtain from the Leibniz rule (72) that

$$a\nabla_r(\theta^p) = \nabla_r(\theta^p)a, \qquad \forall a \in M_n.$$

It follows that elements ω^p_{rq} in the expression (83) are proportional $\mathbf{1} \in M_n$, i.e., complex numbers. Then the relations

$$\nabla_r(\theta^p) = \omega^p_{rq}\theta^q, \qquad \omega^p_{rq} \in \mathbb{C}, \tag{84}$$

define a linear connection on the M_n-bimodule $\mathfrak{O}^1[M_n]$.

Let us consider two examples of linear connections.

(i) Since all derivations of the algebra M_n are inner, we have the curvature-free connection (74) given by the relations

$$\nabla_r(\theta^p) = 0.$$

However, this connection is not torsion-free. The expressions (78) and (82) result in

$$(T\theta^p)(u_r, u_q) = -c^p_{rq}.$$

(ii) One can show that, in matrix geometry, there is a unique torsion-free linear connection

$$\nabla_r(\theta^p) = -c^p_{rq}\theta^q.$$

9 Connes' differential calculus

Connes' differential calculus is based on the notion of a spectral triple [8, 13, 23, 24].

DEFINITION 16. A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is given by a $*$-algebra $\mathcal{A} \subset B(\mathcal{H})$ of bounded operators on a Hilbert space $\mathcal{H}$, together with an (unbounded) self-adjoint operator $D = D^*$ on $\mathcal{H}$ with the following properties:

- the resolvent $(D - \lambda)^{-1}$, $\lambda \neq \mathbb{R}$, is a compact operator on $\mathcal{H}$,

- $[D, \mathcal{A}] \in B(\mathcal{H})$.

$\Box$

The couple $(\mathcal{A}, D)$ is also called a K-cycle over $\mathcal{A}$. In many cases, $\mathcal{H}$ is a $\mathbb{Z}_2$-graded Hilbert space equipped with a projector Γ such that

$$\Gamma D + D\Gamma = 0, \qquad [a, \Gamma] = 0, \quad \forall a \in \mathcal{A},$$

i.e., $\mathcal{A}$ acts on $\mathcal{H}$ by even operators, while D is an odd operator.

Given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, let $(\Omega^*\mathcal{A}, \delta)$ be a universal differential calculus over the algebra $\mathcal{A}$. Let us construct a representation of the graded differential algebra $\Omega^*\mathcal{A}$ by bounded operators on $\mathcal{H}$ when the Chevalley–Eilenberg derivation δ (59) of $\mathcal{A}$ is replaced with the bracket $[D, a]$, $a \in \mathcal{A}$:

$$\begin{aligned}
&\pi : \Omega^*\mathcal{A} \to B(\mathcal{H}),\\
&\pi(a_0\delta a_1 \cdots \delta a_k) \overset{\mathrm{def}}{=} a_0[D, a_1] \cdots [D, a_k].
\end{aligned} \tag{85}$$

Since

$$[D, a]^* = -[D, a^*],$$

we have $\pi(\phi)^* = \pi(\phi^*)$, $\phi \in \Omega^*\mathcal{A}$. At the same time, π (85) fails to be a representation of the graded differential algebra $\Omega^*\mathcal{A}$ because $\pi(\phi) = 0$ does not imply that $\pi(\delta\phi) = 0$. Therefore, one should construct the corresponding quotient in order to obtain a graded differential algebra of operators on $\mathcal{H}$.

Let J_0 be the graded two-sided ideal of $\Omega^*\mathcal{A}$ where

$$J_0^k = \{\phi \in \Omega^k\mathcal{A} \,:\, \pi(\phi) = 0\}.$$

Then it is readily observed that $J = J_0 + \delta J_0$ is a graded differential two-sided ideal of $\Omega^*\mathcal{A}$. By Connes' differential calculus is meant the pair $(\Omega^*_D\mathcal{A}, d)$ such that

$$\begin{aligned}
&\Omega^*_D\mathcal{A} = \Omega^*\mathcal{A}/J,\\
&d[\phi] = [\delta\phi],
\end{aligned}$$

where $[\phi]$ denotes the class of $\phi \in \Omega^*\mathcal{A}$ in $\Omega^*_D\mathcal{A}$. It is a differential calculus over $\Omega^0_D\mathcal{A} = \mathcal{A}$. Its k-cochain submodule $\Omega^*_D\mathcal{A}$ consists of the classes of operators

$$\sum_j a_0^j[D, a_1^j] \cdots [D, a_k^j], \qquad a_i^j \in \mathcal{A},$$

modulo the submodule of operators

$$\{\sum_j [D,b_0^j][D,b_1^j]\cdots[D,b_{k-1}^j]\ :\ \sum_j b_0^j[D,b_1^j]\cdots[D,b_{k-1}^j]=0\}.$$

Let now P be a right finite projective module over the $*$-algebra $\mathcal{A}$. We aim to study a right connection on P with respect to Connes' differential calculus $(\Omega_D^*\mathcal{A},d)$. As was mentioned above in Theorem 14, a right finite projective module has a connection. Let us construct this connection in an explicit form.

Given a generic right finite projective module P over a complex ring $\mathcal{A}$, let

$$\begin{aligned}&\mathbf{p}:\mathbb{C}^N\otimes_C\mathcal{A}\to P,\\ &i_P:P\to\mathbb{C}^N\underset{C}{\otimes}\mathcal{A},\end{aligned}$$

be the corresponding projection and injection, where $\underset{C}{\otimes}$ denotes the tensor product over $\mathbb{C}$. There is the chain of morphisms

$$P\xrightarrow{i_p}\mathbb{C}^N\otimes\mathcal{A}\xrightarrow{\mathrm{Id}\otimes\delta}\mathbb{C}^N\otimes\Omega^1\mathcal{A}\xrightarrow{\mathbf{p}}P\otimes\Omega^1\mathcal{A}, \tag{86}$$

where the canonical module isomorphism

$$\mathbb{C}^N\underset{C}{\otimes}\Omega^1\mathcal{A}=(\mathbb{C}^N\underset{C}{\otimes}\mathcal{A})\otimes\Omega^1\mathcal{A}$$

is used. It is readily observed that the composition (86) denoted briefly as $\mathbf{p}\circ\delta$ is a right universal connection on the module P.

Given the universal connection $\mathbf{p}\circ\delta$ on a right finite projective module P over a $*$-algebra $\mathcal{A}$, let us consider the morphism

$$P\xrightarrow{\mathbf{p}\circ\delta}P\otimes\Omega^1\mathcal{A}\xrightarrow{\mathrm{Id}\otimes\pi}P\otimes\Omega_D^1\mathcal{A}.$$

It is readily observed that this is a right connection ∇_0 on the module P with respect to Connes' differential calculus. Any other right connection ∇ on on P with respect to Connes' differential calculus takes the form

$$\nabla=\nabla_0+\sigma=(\mathrm{Id}\ \otimes\pi)\circ\mathbf{p}\circ\delta+\sigma \tag{87}$$

where σ is an $\mathcal{A}$ module morphism

$$\sigma:P\to P\otimes\Omega_D^1\mathcal{A}.$$

A components σ of the connection ∇ (87) is called a noncommutative gauge field.

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